OBTAINING THE VOLTAGE-STANDING-WAVE RATIO

ON TRANSMISSION LINES

INDEPENDENTLY OF THE DETECTOR CHARACTERISTICS

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ABSTRACT

In. the measurement of impedance at ultra-high frequencies by means of detecting standing waves on transmission lines, it is necessary to know the response law of the detector. This usually involves either a calibration of the detector or an assumption as to its characteristics. By means of the basic transmission-line equations and the general law of detection, expressions are derived which give the voltage-standing-wave ratio as a function of measurable electrical angles, the expressions being independent of the response law of the detector. Special cases of the general equations are discussed, covering applications where high- or low-voltage-standing-wave ratios are to be determined, with or without the use of the voltage maxima or minima. The extension of the methods to the case of attenuating transmission lines is given.

PROBLEM STATUS

AUTHORIZATION

OBTAINING THE VOLTAGE-STANDING-WAVE RATIO ON TRANSMISSION LINES INDEPENDENTLY OF THE DETECTOR CHARACTERISTICS

INTRODUCTION

In the measurement of impedances at ultra-high frequencies by means of detecting standing waves on transmission lines, two quantities uniquely determine the value of the unknown impedance. These are first, the ratio of maximum to minimum voltage on the line, called the voltage-standing-wave ratio, and second the distance from a voltage minimum to the point at which the load is connected. These quantities, in effect permit the complex reflection coefficient of the load to be obtained, the relationships between the various quantities being given below

$$\overline{K} = \frac{\overline{Z}_L - \overline{Z}_0}{\overline{Z}_L + \overline{Z}_0} \tag{1}$$

In equation (1), $\overline{K}$ is the complex reflection coefficient, $\overline{Z}_L$ is the load impedance, and $\overline{Z}_0$ is the characteristic impedance of the transmission line. Equation (1) is solved for the normalized load impedance, giving

$$\frac{\overline{Z}_L}{\overline{Z}_0} = \frac{1 + \overline{K}}{1 - \overline{K}} \tag{2}$$

We can write $\overline{K}$ in polar form as

$$\overline{K} = |\overline{K}| \epsilon^{j\phi} \tag{3}$$

The magnitude of the reflection coefficient is determined from the expression

$$|\overline{K}| = \frac{\frac{V_{max}}{V_{min}} - 1}{\frac{V_{max}}{V_{min}} + 1} = \frac{S_L - 1}{S_L + 1} \tag{4}$$

where V_{max} and V_{min} are respectively the maximum and minimum voltage on the line, and S_L is the voltage-standing-wave ratio.

The phase of the reflection coefficient is determined from the expression

$$\phi = 4\pi \frac{X_{min}}{\lambda} \tag{5}$$

1

where X_{min} is the distance from a voltage minimum to the point at which the load is connected. In Figure 1(a), a voltage distribution such as would exist on a line loaded with an arbitrary impedance, is shown. Shown in Figure 1(b) is a voltage distribution such as would be obtained on a shorted line. In Figure 1(c) are shown several possible detector characteristics, the solid line being the response of a linear detector, the dashed line the response of a square-law detector, and the dotted line the response of a square-root-law detector. The effect of the three detector characteristics shown will be to give the various "indicated" voltage distributions shown in Figures 1(d) and 1(e). Obviously, the values of the voltage-standing-wave ratio which would be obtained from the distributions of Figure 1(d) depend considerably on the characteristics of the detector. If an assumption is made as to the law of the detector, considerable errors can result. Similarly, if a calibration of the detector is made and relied upon at some future time, errors can also result. It is the purpose of this report to derive expressions for the voltage-standing-wave ratio which are independent of the detector characteristics, thereby making detector calibration, or an assumption of detector law, unnecessary.
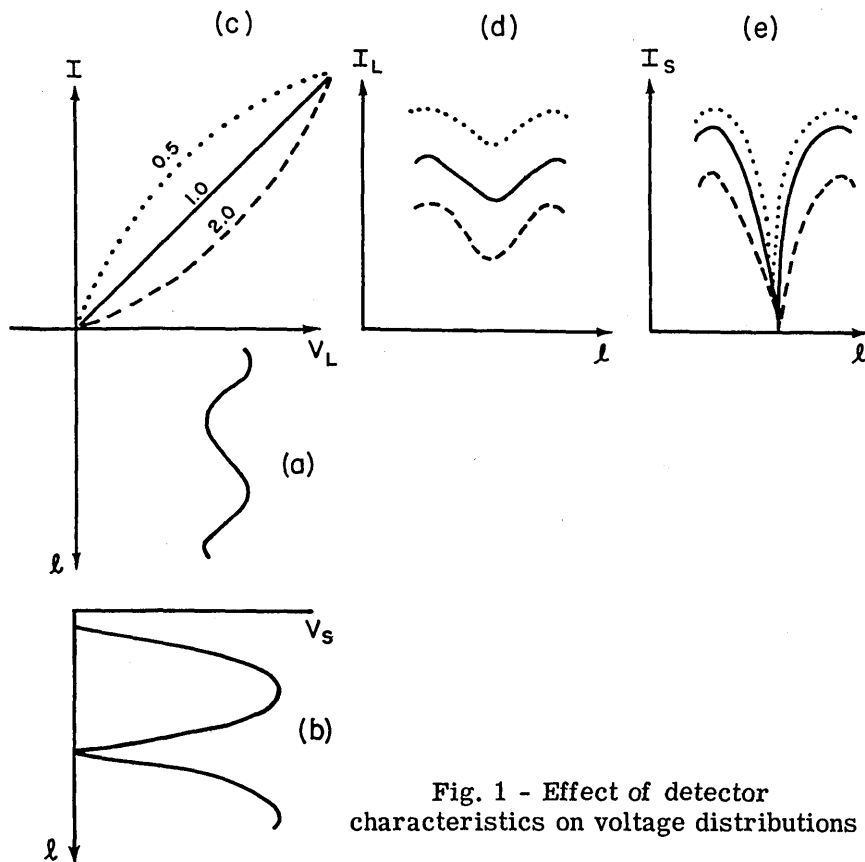

Fig. 1 - Effect of detector characteristics on voltage distributions

METHODS FOR OBTAINING THE VOLTAGE-STANDING-WAVE RATIO ON LOSSLESS TRANSMISSION LINES, ASSUMING A LOSSLESS SHORTING TERMINATION

In Figure 2(a), a transmission line is loaded with an unknown impedance $\bar{Z}_L$. $\bar{V}$ and $\bar{I}$ are the voltage and current respectively at distance 1 from the impedance $\bar{Z}_L$, and $\bar{V}_L$

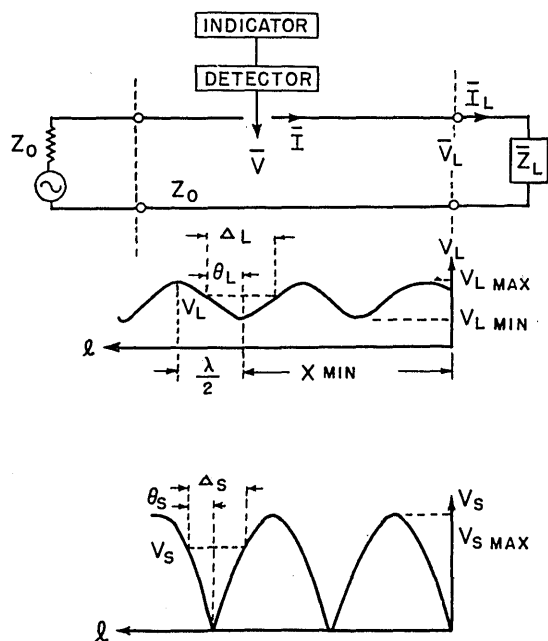Fig. 2 - Voltage distributions of loaded and shorted transmission lines

and $\overline{I}_L$ are the load voltage and current. The familiar transmission-line equations in hyperbolic form are

$$\overline{V} = \overline{V}_L \cosh \overline{\gamma}l + \overline{I}_L\, Z_0 \sinh \overline{\gamma}l \quad (6)$$

$$\overline{I} = \overline{I}_L \cosh \overline{\gamma}l + \frac{\overline{V}_L}{Z_0}\, \sinh \overline{\gamma}l \quad (7)$$

$\overline{\gamma}$ is the complex propagation constant and is equal to

$$\overline{\gamma} = \alpha + j\beta \quad (8)$$

where α is the attenuation per unit length and β is the phase shift per unit length given by

$$\beta = \frac{2\pi}{\lambda} \quad (9)$$

λ being the wavelength in the transmission line.

If the attenuation is negligible we get

$$\cosh \overline{\gamma}l = \cosh j\beta l = \cos \beta l \quad (10)$$

$$\sinh \overline{\gamma}l = \sinh j\beta l = j \sin \beta l \quad (11)$$

Equations (6) and (7) now become

$$\overline{V} = \overline{V}_L \cos \beta l + j\,\overline{I}_L\, Z_0\, \sin \beta l \quad (12)$$

$$\overline{I} = \overline{I}_L \cos \beta l + j\, \frac{\overline{V}_L}{Z_0}\, \sin \beta l \quad (13)$$

At a distance X_{min} from the load we will have a voltage minimum. At this same point the current will be a maximum. The impedance at this point is a pure resistance so that V_{min} and I_{max} will be in phase. In equation (12) if we replace $\overline{V}_L$ by V_{min} and $\overline{I}_L$ by I_{max}, the voltage $\overline{V}$ will now be the voltage at distance l from the voltage minimum, equation (12) is now

$$\overrightarrow{V} = V_{min} \cos \beta l + j\, I_{max}\, Z_0\, \sin \beta l \quad (14)$$

The absolute value of equation (14) is given by

$$V^2 = V^2_{min} \cos^2 \beta l + I^2_{max}\, Z_0^2 \sin^2 \beta l \quad (15)$$

If we let $l = 0$ in equation (15) we get $V = V_{min}$ which checks that equation (15) refers the voltage V to a minimum point.

We know that at a point a quarter-wavelength from a voltage minimum, a voltage maximum will exist, so that letting $l = \lambda/4$ in equation (15) gives

$$V = V_{max} = I_{max}\, Z_0 \quad (16)$$

Substituting for I_{max} in equation (15) gives

$$V^2 = V^2_{min} \cos^2 \beta 1 + V^2_{max} \sin^2 \beta 1 \tag{17}$$

Equation (17) now gives a convenient expression for the actual voltage distribution on a line loaded with an arbitrary impedance as shown in Figure 2(b). We can let

$$\theta = \beta 1 = 2\pi \frac{1}{\lambda} \tag{18}$$

For convenience we let

$$\Delta = 2\,1 \tag{18}$$

Equation (18) then becomes

$$\theta = \pi \frac{\Delta}{\lambda} \tag{19}$$

We can therefore write for the voltage distribution of Figure 2(b)

$$V^2_L = V^2_{L\,max} \sin^2 \theta_L + V^2_{L\,min} \cos^2 \theta_L \tag{20}$$

For a shorted line, $V_{S\,min} = 0$, provided the shorting termination is lossless, and the analytic expression for the voltage distribution on a shorted line becomes

$$V^2_S = V^2_{s\,max} \sin^2 \theta_s \tag{21}$$

In Figure 3(a), is shown a voltage distribution such as would actually exist on a loaded line. For two values of voltage V_{L_1} and V_{L_2} we can write equation (20) in terms of the half angular widths θ_{L_1} and θ_{L_2} as follows

$$V^2_{L_2} = V^2_{L\,max} \sin^2 \theta_{L_2} + V^2_{L\,min} \cos^2 \theta_{L_2} \tag{22}$$

$$V^2_{L_1} = V^2_{L\,max} \sin^2 \theta_{L_1} + V^2_{L\,min} \cos^2 \theta_{L_1} \tag{23}$$

Similarly, for the voltage distribution on a shorted line, shown in Figure 3(b) we can write

$$V^2_{S_2} = V^2_{s\,max} \sin^2 \theta_{S2} \tag{24}$$

$$V^2_{S_1} = V^2_{s\,max} \sin^2 \theta_{S1} \tag{25}$$

As before, we have shown in Figure 3(c) a linear detector characteristic (solid curve) and a non-linear characteristic (dashed curve), with Figures 3(d) and 3(e) showing the true voltage distributions (solid curves) which would be obtained from the linear detector and the "indicated" voltage distributions (dashed curves) which would be obtained from the non-linear detector.

At a regular point of the non-linear characteristic of Figure 3(c) we can write a Taylor's series

$$V = f(I_0) + \frac{df}{dI}(\Delta\,I) + \frac{1}{2} \frac{d^2 f}{dI^2}(\Delta I)^2 + \ldots\ldots \tag{26}$$

where I_0 is the operating point at which the derivatives are evaluated. We can thus write
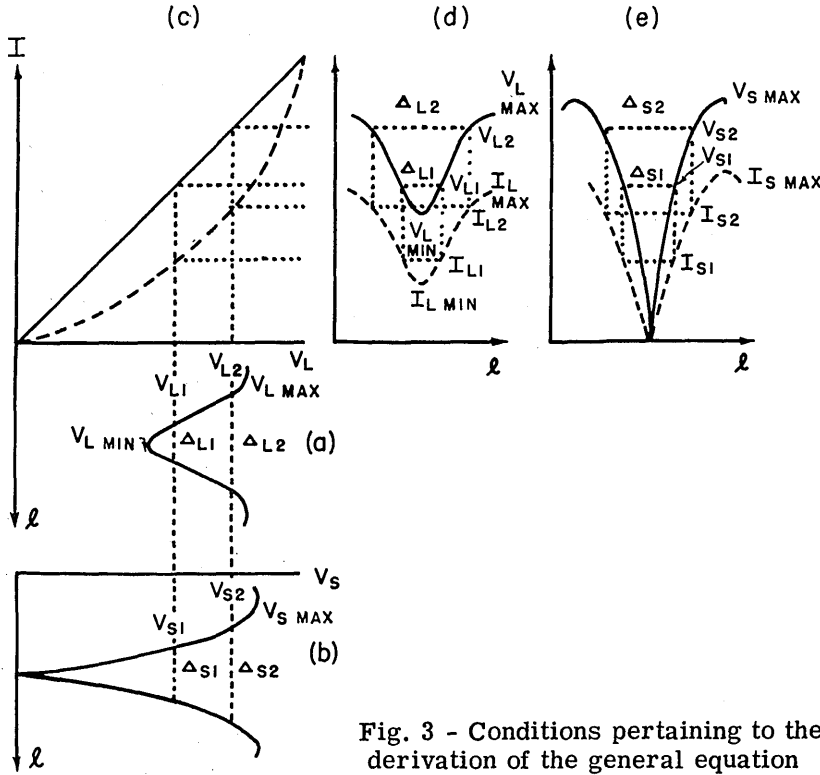
$$V^2 = f(I) \tag{27}$$

Fig. 3 - Conditions pertaining to the
derivation of the general equation

the particular form of the function depending only on the operating point. We can thus substitute for V^2 in equations (22) thru (25) giving

$$V_{L\,max}^2 \sin^2\theta_{L_2} + V_{L\,min}^2 \cos^2\theta_{L_2} = f(I_{L_2}) \tag{28}$$

$$V_{L\,max}^2 \sin^2\theta_{L_1} + V_{L\,min}^2 \cos^2\theta_{L_1} = f(I_{L_1}) \tag{29}$$

$$V_{s\,max}^2 \sin^2\theta_{s2} = f(I_{s2}) \tag{30}$$

$$V_{s\,max}^2 \sin^2\theta_{s1} = f(I_{s1}) \tag{31}$$

In effect, the substitution by means of equation (27) transforms the true voltage distributions of Figures 3(d) and 3(e) into the "indicated" voltage distributions, the angles θ_{L_2}, θ_{L_1}, θ_{s_2}, θ_{s_1}, remaining unchanged during the transformation. Equations (28) thru (31) thus give expressions for the "indicated" voltage distributions on the line. If now we make $V_{L_2} = V_{s_2}$, then $I_{L_2} = I_{s_2}$, so that f $(I_{L_2}) = f(I_{s_2})$ and equation (28) becomes equal to equation (30). Similarly, if we make $V_{L_1} = V_{s_1}$, $I_{L_1} = I_{s_1}$, and f $(I_{L_1}) = f(I_{s_1})$ making it possible to equate equations (29) and (31). It is of course not necessary for the functions at the two voltage levels to be equal to each other. We can thus combine these four equations (28) thru (31) to give

$$\frac{V_{s\,max}^2 \sin^2\theta_{s2}}{V_{s\,max}^2 \sin^2\theta_{s1}} = \frac{V_{L\,max}^2 \sin^2\theta_{L2} + V_{L\,min}^2 \cos^2\theta_{L2}}{V_{L\,max}^2 \sin^2\theta_{L1} + V_{L\,min}^2 \cos^2\theta_{L1}} \tag{32}$$

It will be noticed that the quantity $V_{s\,max}$ cancels from equation (32). Recalling that the

voltage-standing-wave ratio S_L is given by

$$S_L = V_{L\,max}/V_{L\,min} \qquad (4)$$

equation (32) becomes

$$\frac{\sin^2\theta_{S2}}{\sin^2\theta_{S1}} = \frac{S_L^2 \sin^2\theta_{L_2} + \cos^2\theta_{L_2}}{S_L^2 \sin^2\theta_{L_1} + \cos^2\theta_{L_1}} \qquad (33)$$

Solving for S_L^2 gives

$$S_L^2 = \frac{\dfrac{\sin^2\theta_{S2}}{\sin^2\theta_{S1}}\cos^2\theta_{L_1} - \cos^2\theta_{L_2}}{\sin^2\theta_{L_2} - \dfrac{\sin^2\theta_{S2}}{\sin^2\theta_{S1}}\sin^2\theta_{L_1}} \qquad (34)$$

Equation (34) thus gives the voltage-standing-wave ratio as a function of four measurable electrical angles, the expression being independent of the detector characteristics at each of the two voltage levels at which the angles were measured. This general expression can be applied in the following manner.

Method 1

The line is loaded and two convenient values of "indicated" voltage are chosen. These are designated I_{L_2} and I_{L_1} in Figure 4(a). The positions on the loaded line at which $I_L = I_{L_2}$ and $I_L = I_{L_1}$ are noted thus permitting the quantities Δ_{L_2} and Δ_{L_1} to be obtained by subtraction. The angles θ_{L_2} and θ_{L_1} can then be calculated by means of equation (19). The line is then shorted, the maximum voltage being made greater than that of I_{L_2} in the loaded case. The quantities Δ_{S_2} and Δ_{S_1} are determined at the voltage levels at which $I_{S_2} = I_{L_2}$ and $I_{S_1} = I_{L_1}$, and the angles θ_{S_2} and θ_{S_1} are calculated. Equation (34) then permits the calculation of the voltage-standing-wave ratio without using either the voltage maxima or minima on the loaded line. In this general form the equation cannot be readily plotted but it can be simplified in various ways which will be discussed below.

Figure 4
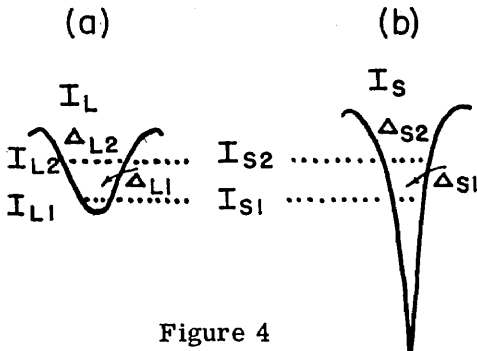
In measuring high voltage-standing-wave ratios it is usually not possible to get both the maximum and minimum voltage on the same scale of the indicator. Thus if an observable reading of the minimum voltage is obtained, the maximum voltage will be off scale. If the maximum voltage were set on scale the minimum would not be observable. Measurements of high voltage-standing-wave ratios are most conveniently and accurately made in the vicinity of a voltage minimum, the voltage maximum not being used. For this reason, methods which are to be discussed will be divided into two groups: those which do not require the use of the voltage maxima on the loaded line will be indicated as most useful for the measurement of high voltage-standing-wave ratios, and those which use the voltage maxima on the loaded line will be indicated as most useful for the measurement of low voltage-standing-wave ratios. As an additional subdivision, methods which do not require the use of voltage minima on the loaded line will be indicated as most useful when noise obscures the minima. Noise will, in most practical cases, cause the most difficulty when high VSWR's are to be measured, but it is possible that owing to the lack of sufficient

power during the measurement of a low VSWR, noise would obscure the minima. We will thus also discuss special cases of equation (34) applicable to the measurement of low VSWR when noise obscures the voltage minima on the line.

Method 2

In this method (Figure 5) the values Δ_{L_2} and Δ_{L_1} are determined as before but when the line is shorted the maximum is made equal to I_{L_2}. This makes $\Delta_{S_2} = \lambda/2$ so that $\theta_{S_2} = 90°$. For $\theta_{S_2} = 90°$, equation (34) reduces to

$$S_L{}^2 = \frac{\dfrac{\cos^2\theta_{L_1}}{\sin^2\theta_{S_1}} - \cos^2\theta_{L_2}}{\sin^2\theta_{L_2} - \dfrac{\sin^2\theta_{L_1}}{\sin^2\theta_{S_1}}} \qquad (35)$$
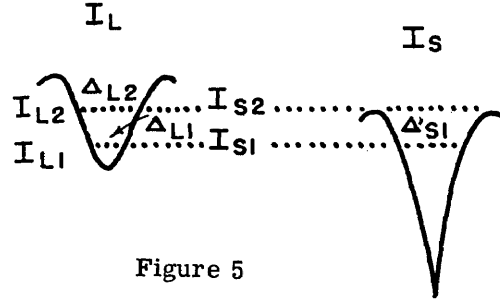
Figure 5

This equation can be used to determine a high VSWR when noise obscures the voltage minima on the line, since neither the maximum or minimum voltage on the loaded line is used. For convenience in plotting equation (35) we make the following simplifying assumptions:

$$\theta_{L_2} < 10° \qquad (36)$$

$$\cos\theta_{L_2} \approx 1$$
$$\cos\theta_{L_1} \approx 1$$
$$\sin\theta_{L_2} \approx \pi\Delta_{L_2}/\lambda \qquad (37)$$
$$\sin\theta_{L_1} \approx \pi\Delta_{L_1}/\lambda$$

If we further let

$$\Delta_{L_2} = \sqrt{2}\,\Delta_{L_1} \qquad (38)$$

Equation (35) reduces to

$$S_L^2\left(\frac{\pi\Delta_{L_1}}{\lambda}\right)^2 = \frac{\cos^2\theta_{S_1}}{2\sin^2\theta_{S_1} - 1} \qquad (38)$$

which can be written as

$$S_L^2\left(\frac{\pi\Delta_{L_1}}{\lambda}\right)^2 = \frac{1}{\tan^2\theta_{S_1} - 1} \qquad (39)$$

Since $\theta_{L_2} = \sqrt{2}\theta_{L_1}$, using equation (36) the condition imposed on equation (38) is that

$$\theta_{L_1} < 7° \qquad (40)$$

From equation (39) it is also evident that

$$\theta_{S_1} < 45° \qquad (41)$$

A nomograph of equation (39) is shown in Plate 1 on page 16. Instead of S_L we have plotted $db_L = 20 \log_{10} S_L$ on the appropriate scale.

Method 3

This method and the one following are included to complete the discussion of the possible applications of equation (34). Referring to Figure 6 we determine Δ_{S_2} at the voltage level $I_{S_2} = I_{L\,max}$ thereby making $\theta_{L_2} = 90^0$ equation (34) becoming

$$S_L^2 = \frac{\sin^2 \theta_{S_2} \cos^2 \theta_{L_1}}{\sin^2 \theta_{S_1} - \sin^2 \theta_{S_2} \sin^2 \theta_{L_1}} \tag{42}$$

This method could be used for measuring low values of VSWR when noise obscures the minima.
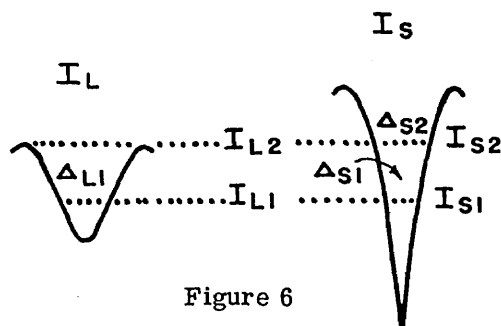
Figure 6
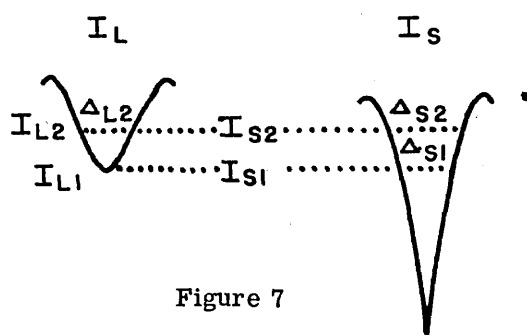
Figure 7

Method 4

Referring to Figure 7, we measure Δ_{S_1} at the voltage level $I_{S_1} = I_{L\,min}$ thereby making $\theta_{L_1} = 0^0$, equation (34) becoming

$$S_L^2 = \frac{\dfrac{\sin^2 \theta_{S_2}}{\sin^2 \theta_{S_1}} - \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_2}} \tag{43}$$

This method can be used for measuring high values of VSWR.

Method 5

In this method (Figure 8) Δ_{L_2} is measured at some convenient level I_{L_2} with the line loaded. When the line is shorted the value of $I_{S\,max}$ is made equal to I_{L_2} and Δ_{S_1} is measured at the voltage level at which $I_S = I_{L\,min}$. This procedure makes $\theta_{L_1} = 0^0$ and $\theta_{S_2} = 90^0$, equation (34) becoming

$$S_L^2 - 1 = \frac{\cot^2 \theta_{S_1}}{\sin^2 \theta_{L_2}} \tag{44}$$

For $S_L > 10$, equation (14) becomes

$$S_L = \frac{\cot \theta_{S_1}}{\sin \theta_{L_2}} \qquad S_L > 10 \tag{45}$$

A nomograph of equation (45) is shown in Plate 2 on page 17. This method is the one most useful for the measurement of high values of VSWR.
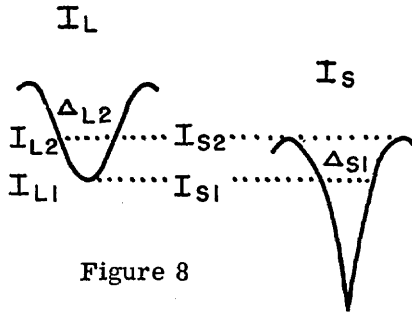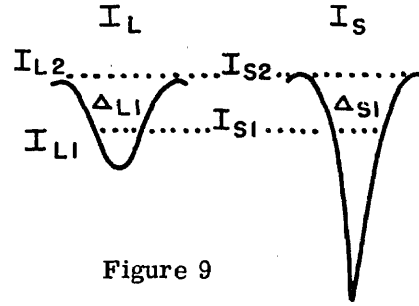

Figure 8

Figure 9

Method 6

Here, in Figure 9, the maxima on the loaded and shorted lines are made equal and Δ_{L_1} and Δ_{S_1} determined at some convenient level $I_{L_1} = I_{S_1}$. This procedure makes $\theta_{L_2} = \theta_{S_2} = 90^\circ$ with equation (34) becoming

$$S_L^2 = \frac{\cos^2 \theta_{L_1}}{\sin^2 \theta_{S_1} - \sin^2 \theta_{L_1}} \qquad (46)$$

A nomograph of equation (46) is shown in Plate 3 on page 18. This method is the one most useful for the measurement of low values of VSWR when noise obscures the minima on the loaded line.
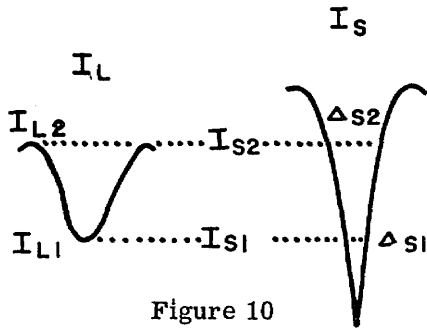
Figure 10

Method 7

In this method (Figure 10) Δ_{S_2} is determined at the voltage level $I_{S_2} = I_{L\,max}$ and Δ_{S_1} is determined at the level $I_{S_1} = I_{L\,min}$. We thus have $\theta_{L_1} = 0^\circ$ and $\theta_{L_2} = 90^\circ$ equation (34) becoming

$$S_L = \frac{\sin \theta_{S_2}}{\sin \theta_{S_1}} \qquad (47)$$

This method could be used for the measurement of low values of VSWR.

Method 8

In this method (Figure 11) the maxima in both the loaded and shorted lines are made equal and Δ_{S_1} is determined at the voltage level $I_{S_1} = I_{L\,min}$. This makes $\theta_{L_2} = \theta_{S_2} = 90^\circ$ and $\theta_{L_1} = 0^\circ$, equation (34) becoming
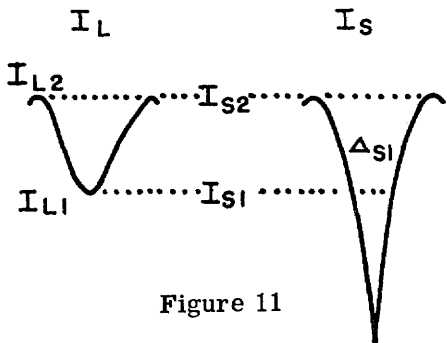

Figure 11

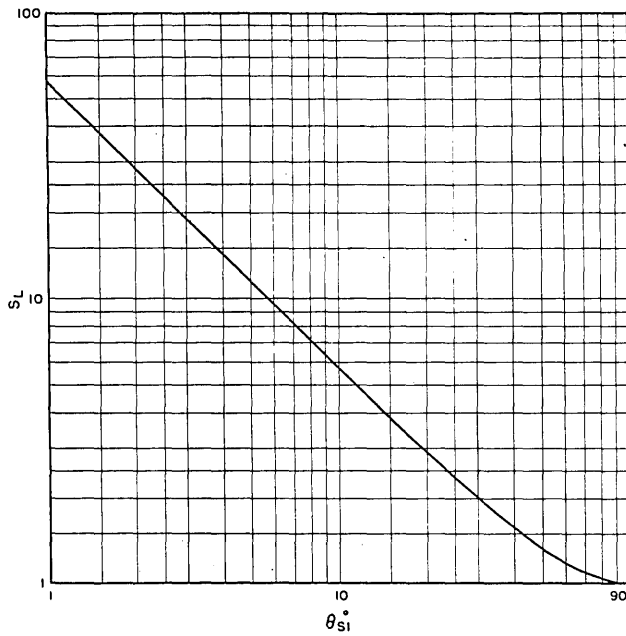$$S_L = \frac{1}{\sin \theta_{S_1}} \qquad (48)$$

Fig. 12 - Graph for method 8

A graph of equation (48) is shown in Figure 12. This method is the one most useful for the measurement of low values of VSWR.

Expression for the Law of the Detector

An expression for the law of the detector can be obtained by making the assumption that the response of the detector has the general form

$$I = k V^{n} \qquad (49)$$

This not only assumes that detector response behaves the above equation at any point but also that the law of the detector is the same over the entire range of V.

Assuming that equation (49) applies we can solve it for V giving

$$V = \left(\frac{I}{k}\right)^{1/n} \qquad (50)$$

We had as the general expressions for two values of voltage V_{S_2} and V_{S_1} on a shorted line

$$V_{S_2}^{2} = V_{s\,max}^{2} \sin^{2}\theta_{S_2} \qquad (24)$$

$$V_{S_1}^{2} = V_{s\,max}^{2} \sin^{2}\theta_{S_1} \qquad (25)$$

From equation (50) we can write

$$V_{S_2}^{2} = \left(\frac{I_{S2}}{k}\right)^{2/n} \qquad (51)$$

$$V_{S_1}^{2} = \left(\frac{I_{S1}}{k}\right)^{2/n} \qquad (52)$$

Substituting from equations (51) and (52) into (24) and (25) gives

$$\left(\frac{I_{S2}}{k}\right)^{2/n} = V_{s\,max}^{2} \sin^{2}\theta_{S_2} \qquad (53)$$

$$\left(\frac{I_{S1}}{k}\right)^{2/n} = V_{s\,max}^{2} \sin^{2}\theta_{S_1} \qquad (54)$$

Dividing equation (53) by (54) gives

$$\left(\frac{I_{S2}}{I_{S1}}\right)^{1/n} = \frac{\sin\theta_{S_2}}{\sin\theta_{S_1}} \qquad (55)$$

Solving for n

$$n = \frac{\log_{10}(I_{S2}/I_{S1})}{\log_{10}(\sin\theta_{S_2}/\sin\theta_{S_1})} \qquad (56)$$

Equation (56) can be used to determine the law of the detector n, for methods 1 through 8; but for methods 2, 5, 6, and 8 for which the applicable equations have been plotted, $\theta_{S_2} = 90^o$ and equation (56) can be written

$$n = \frac{\log_{10}(I_2/I_1)}{\log_{10}(1/\sin\theta_{S_1})} \qquad (57)$$

A nomograph of equation (57) is shown in Plate 4 on page 19.

METHODS FOR OBTAINING THE VOLTAGE-STANDING-WAVE RATIO ON LOSSLESS TRANSMISSION LINES INDEPENDENTLY OF SHORTED LINE CONDITIONS

We now describe several methods by which the VSWR can be obtained by using two different voltage distributions on the loaded line. Referring to Figure 13(a) we have a voltage distribution I_L on the line. Equations (28) and (29) apply as was previously the case

$$V_{L\,max}^2 \sin^2\theta_{L2} + V_{L\,min}^2 \cos^2\theta_{L2} = f(I_{L2}) \qquad (28)$$

$$V_{L\,max}^2 \sin^2\theta_{L1} + V_{L\,min}^2 \cos^2\theta_{L1} = f(I_{L1}) \qquad (29)$$

(a)   (b)

If we adjust the power input to the line so that a somewhat lower distribution I_L' is obtained as shown in Figure 13(b), equations (28) and (29) can again be applied as follows

$$V_{L\,max}'^2 \sin^2\theta_{L2}' + V_{L\,min}'^2 \cos^2\theta_{L2}' = f(I_{L2}') \qquad (58)$$

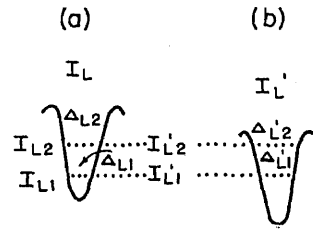$$V_{L\,max}'^2 \sin^2\theta_{L1}' + V_{L\,min}'^2 \cos^2\theta_{L1}' = f(I_{L1}') \qquad (59)$$

Figure 13

If now we let $I_{L2} = I_{L2}'$, then $f(I_{L2}) = f(I_{L2}')$ and equation (28) equals equation (58). Similarly if we let $I_{L1} = I_{L1}'$, then $f(I_{L1}) = f(I_{L1}')$ and equation (29) equals equation (59). We can combine these four equations giving

$$\frac{V_{L\,max}^2 \sin^2\theta_{L2} + V_{L\,min}^2 \cos^2\theta_{L2}}{V_{L\,max}^2 \sin^2\theta_{L1} + V_{L\,min}^2 \cos^2\theta_{L1}} = \frac{V_{L\,max}'^2 \sin^2\theta_{L2}' + V_{L\,min}'^2 \cos^2\theta_{L2}'}{V_{L\,max}'^2 \sin^2\theta_{L1}' + V_{L\,min}'^2 \cos^2\theta_{L1}'} \qquad (60)$$

We know that

$$S_L = \frac{V_{L\,max}}{V_{L\,min}} = \frac{V_{L\,max}'}{V_{L\,min}'} \qquad (61)$$

Equation (60) becomes

$$\frac{S_L^2 \sin^2\theta_{L2} + \cos^2\theta_{L2}}{S_L^2 \sin^2\theta_{L1} + \cos^2\theta_{L1}} = \frac{S_L^2 \sin^2\theta_{L2}' + \cos^2\theta_{L2}'}{S_L^2 \sin^2\theta_{L1}' + \cos^2\theta_{L1}'} \qquad (62)$$

Solving for S_L^2 gives

$$S_L^2 = \frac{\cos^2\theta_{L1}' \cos^2\theta_{L2} - \cos^2\theta_{L2}' \cos^2\theta_{L1}}{\sin^2\theta_{L1}' \sin^2\theta_{L2} - \sin^2\theta_{L2}' \sin^2\theta_{L1}} \qquad (63)$$

Two applications of this equation will be discussed.

Method 9

We have here made $\theta_{L'_2} = 90^0$ giving for equation (63)

$$S_L^2 = \frac{\cos^2\theta_{L_1}' \cos^2\theta_{L_2}}{\sin^2\theta_{L_1}' \sin^2\theta_{L_2} - \sin^2\theta_{L_1}}$$ (64)

This method (Figure 14) could be used for obtaining large values of VSWR when noise obscures the voltage minima on the loaded line.

Method 10

Figure 14

In this case (Figure 15) we have set the conditions of measurement such that $\theta_{L_2} = 90^0$ and $\theta_{L_1} = 0^0$ giving for equation (63)

$$S_L = \frac{\cot\theta_{L_1}'}{\tan\theta_{L_2}}$$ (65)

Equation (65) can be applied for all ranges of VSWR however it is most useful for high values. Shown in Plate 5 on page 20 is a nomograph of equation (65).

Expression for the Law of the Detector

Figure 15

Using a method of analysis similar to that used before we obtain for the Method 10 the following expression for the law of the detector

$$n = \frac{\log_{10}(I_2/I_1)}{\log_{10}(\cos\theta_{L_2}/\sin\theta_{L_1}')}$$ (66)

METHODS FOR OBTAINING THE VOLTAGE-STANDING-WAVE RATIO ON LOSSLESS TRANSMISSION LINES INCLUDING EFFECT OF LOSSY SHORTING TERMINATION

If the short circuit is lossy, the minimum voltage on the shorted line will not be equal to zero, the conditions of Figure 16(b) being obtained. We can again write in the loaded case of Figure 16(a).
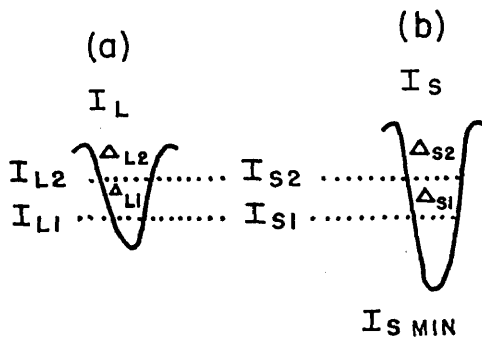
Figure 16

$$V_{L\max}^2 \sin^2\theta_{L_2} + V_{L\min}^2 \cos^2\theta_{L_2} = f(I_{L_2})$$ (67)

$$V_{L\max}^2 \sin^2\theta_{L_1} + V_{L\min}^2 \cos^2\theta_{L_1} = f(I_{L_1})$$ (68)

and in the shorted case of Figure 16(b)

$$V_{s\max}^2 \sin^2\theta_{s_2} + V_{s\min}^2 \cos^2\theta_{s_2} = f(I_{s_2})$$ (69)

$$V_{s\max}^2 \sin^2\theta_{s_1} + V_{s\min}^2 \cos^2\theta_{s_1} = f(I_{s_1})$$ (70)

By making $I_{L_2} = I_{S_2}$ and $I_{L_1} = I_{S_1}$, we can combine the above four equations giving

$$\frac{V^2_{L\,max}\,\sin^2\theta_{L_2} + V^2_{L\,min}\,\cos^2\theta_{L_2}}{V^2_{L\,max}\,\sin^2\theta_{L_1} + V^2_{L\,min}\,\cos^2\theta_{L_1}} = \frac{V^2_{s\,max}\,\sin^2\theta_{s_2} + V^2_{s\,min}\,\cos^2\theta_{s_2}}{V^2_{s\,max}\,\sin^2\theta_{s_1} + V^2_{s\,min}\,\cos^2\theta_{s_1}} \tag{71}$$

letting

$$S_L = \frac{V_{L\,max}}{V_{L\,min}} \tag{72}$$

and

$$S_S = \frac{V_{s\,max}}{V_{s\,min}} \tag{73}$$

equation (71) becomes

$$\frac{S^2_L\,\sin^2\theta_{L_2} + \cos^2\theta_{L_2}}{S^2_L\,\sin^2\theta_{L_1} + \cos^2\theta_{L_1}} = \frac{S^2_S\,\sin^2\theta_{s_2} + \cos^2\theta_{s_2}}{S^2_S\,\sin^2\theta_{s_1} + \cos^2\theta_{s_1}} \tag{74}$$

Solving for S^2_L gives

$$S^2_L = \frac{Q\,\cos^2\theta_{L_1} - \cos^2\theta_{L_2}}{\sin^2\theta_{L_1} - Q\,\sin^2\theta_{L_1}} \tag{75}$$

$$Q = \frac{\sin^2\theta_{s_2} + \dfrac{\cos^2\theta_{s_2}}{S^2_S}}{\sin^2\theta_{s_1} + \dfrac{\cos^2\theta_{s_1}}{S^2_S}} \tag{76}$$

As before, we consider as practical cases those in which $\theta_{s_2} = 90^0$, the conditions of Figures 17(a) and (b) being obtained, equation (76) becoming

$$Q = \frac{1}{\sin^2\theta_{s_1} + \dfrac{\cos^2\theta_{s_1}}{S^2_S}} \tag{77}$$

To evaluate S_L by means of equations (75) and (77) we need the value of the VSWR on the shorted line. This can be obtained by applying Method 10 referred to on page 12. Referring to Figure 17(c), the line is shorted and Δ_{s_2}' is measured at voltage level I_{s_2}'. The power into the shorted line is reduced till $I_{s\,max} = I_{s_2}'$ as shown in Figure 17(b). The value of S_S is then given by equation (65)

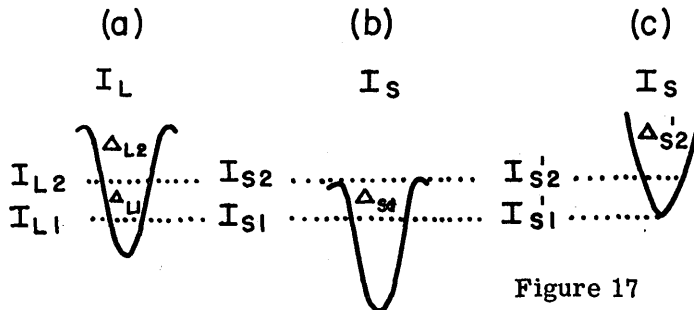$$S_S = \frac{\cot\theta_{s_1}}{\tan\theta_{s_2}'} \tag{78}$$

Figure 17

Substituting S_S from equation (78) into equation (77) gives

$$Q = \frac{\cos^2 \theta'_{S_2}}{\sin^2 \theta_{S_1}} \tag{79}$$

Substituting Q from equation (79) into equation (75) gives

$$S_L^2 = \frac{\dfrac{\cos^2 \theta'_{S_2}}{\sin^2 \theta_{S_1}} \cos^2 \theta_{L_1} - \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_2} - \dfrac{\cos^2 \theta'_{S_2}}{\sin^2 \theta_{S_1}} \sin^2 \theta_{L_1}} \tag{80}$$

Equation (80) can now be used to determine values of VSWR when noise obscures the minimum as shown in Figure 17.

Several other applications of equation (80) will now be discussed.
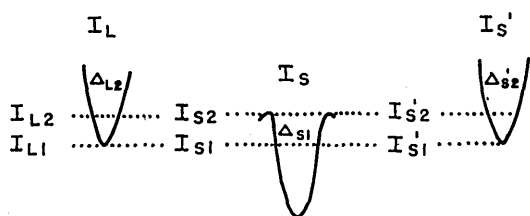
Figure 18

Figure 19

Figure 20

Method 11

In this method (Figure 18) we let $\theta_{L_1} = 0°$, equation (80) becoming

$$S_L^2 = \frac{\dfrac{\cos^2 \theta'_{S_2}}{\sin^2 \theta_{S_1}} - \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_2}} \tag{81}$$

This case could be used for measurements of high values of VSWR.
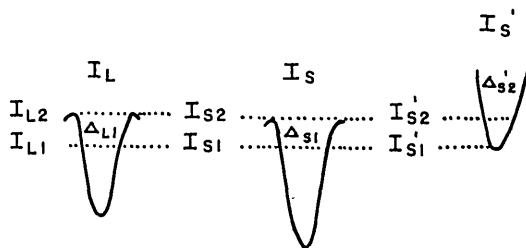
Method 12

Here we let $\theta_{L_2} = 90°$, equation (80) becoming

$$S_L^2 = \frac{\cos^2 \theta_{L_1}}{\dfrac{\sin^2 \theta_{S_1}}{\cos^2 \theta'_{S_2}} - \sin^2 \theta_{L_1}} \tag{82}$$

This method (Figure 19) can be used for low values of VSWR when noise obscures the minima.
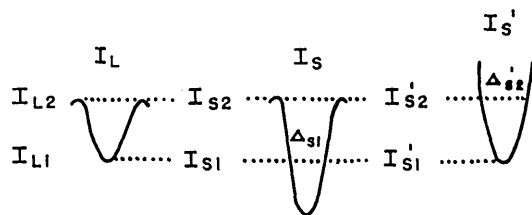
Method 13

In this method (Figure 20) we let $\theta_{L_2} = 90°$, and $\theta_{L_1} = 0°$. equation (80) becoming

$$S_L = \frac{\cos \theta'_{S_2}}{\sin \theta_{S_1}} \tag{83}$$

A nomograph of equation (83) is shown in Plate 6 on page 21. This method is a useful one for low values of VSWR.

In practice, the voltage distributions of Figures 17 to 20 would be obtained as follows. First, the transmission line is shorted and sufficient power fed to the line to enable $I_{S'min}$ to be read on the indicator. For some convenient value of I_{S_2}', Δ_{S_2}' would be measured. The power input to the line would then be lowered until $I_{S\,max} = I_{S_2}'$. The quantity Δ_{S_1} would then be measured at the voltage level $I_{S_1} = I_{S'min}$. The line would then be loaded with the unknown impedance and the input power adjusted so that $I_{L\,min} = I_{S'min}$ (Figure 18) or $I_{L\,max} = I_{S\,max}$ (Figures 19 and 20). The quantity Δ_L could then be measured in those cases in which it is required.

Expression for the Law of the Detector

Using the same method of analysis as before we obtain for the methods illustrated by Figures 17 through 20 the following expression for the law of the detector.

$$n = \frac{\log_{10}\,(I_2/I_1)}{\log_{10}\,(\cos\theta_{S_2}'/\sin\theta_{S_1})} \tag{84}$$

SUMMARY OF METHODS

For convenience in applying the methods derived in this report the significant ones are assembled in Table I on page 25. It will be noted that the expressions obtained in the cases where the effect of the lossy short is taken into account reduce to those obtained assuming a lossless short when $\cos\theta_{S_2}' = 1$. In making measurements using the distribution on shorted lines initial measurements will enable one to determine whether this approximation can be made. In contrast with the Methods requiring the use of distributions on shorted lines, Methods 8, 9, and 10 would appear to be the most useful.

CONCLUSIONS

Expressions have been derived which do not appear to have been previously available enabling one to measure the VSWR on transmission lines independently of the detector characteristics. It is felt that these expressions will be of particular usefulness when crystals are used as detectors, since these are not always square law and may not retain their calibration for any appreciable length of time. In using other detectors the methods indicated here have the advantage of not requiring any assumption as to the law of the detector.
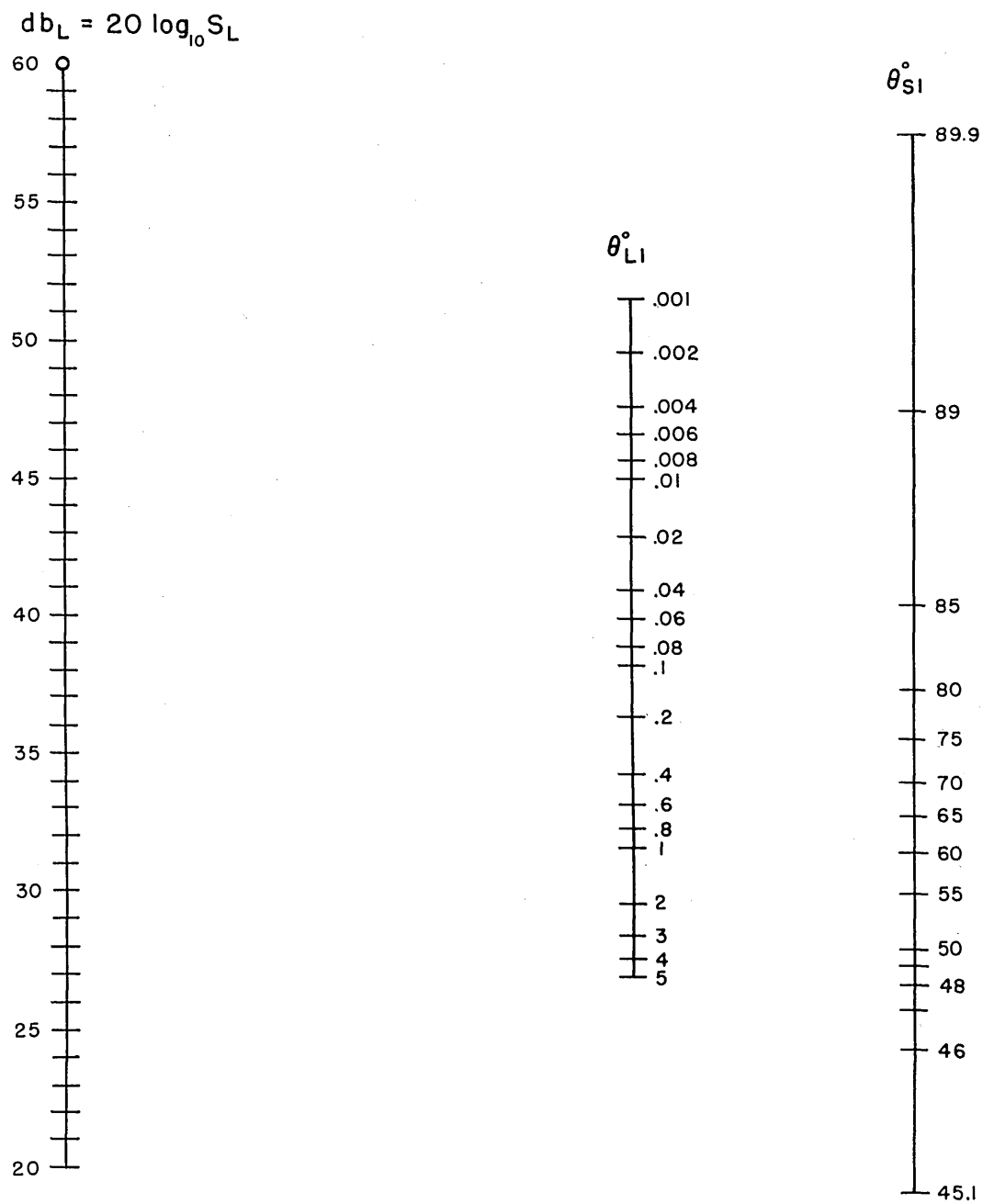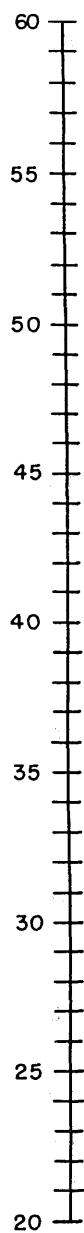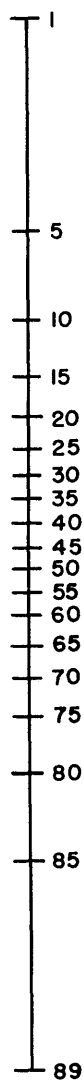
* * *

$db_L = 20 \log_{10} S_L$

60
55
50
45
40
35
30
25
20

θ°_{L1}

.001
.002
.004
.006
.008
.01
.02
.04
.06
.08
.1
.2
.4
.6
.8
1
2
3
4
5

θ°_{S1}

89.9
89
85
80
75
70
65
60
55
50
48
46
45.1

Plate 1

Nomograph for Method 2; $\theta_{L_2} = \sqrt{2}\, \theta_{L_1}$

$$db_L = 20 \log_{10} S_L$$

θ°_{S1}

θ°_{L2}
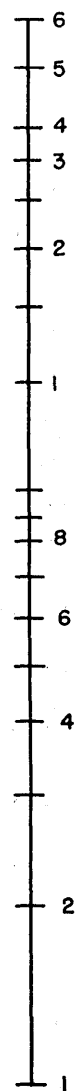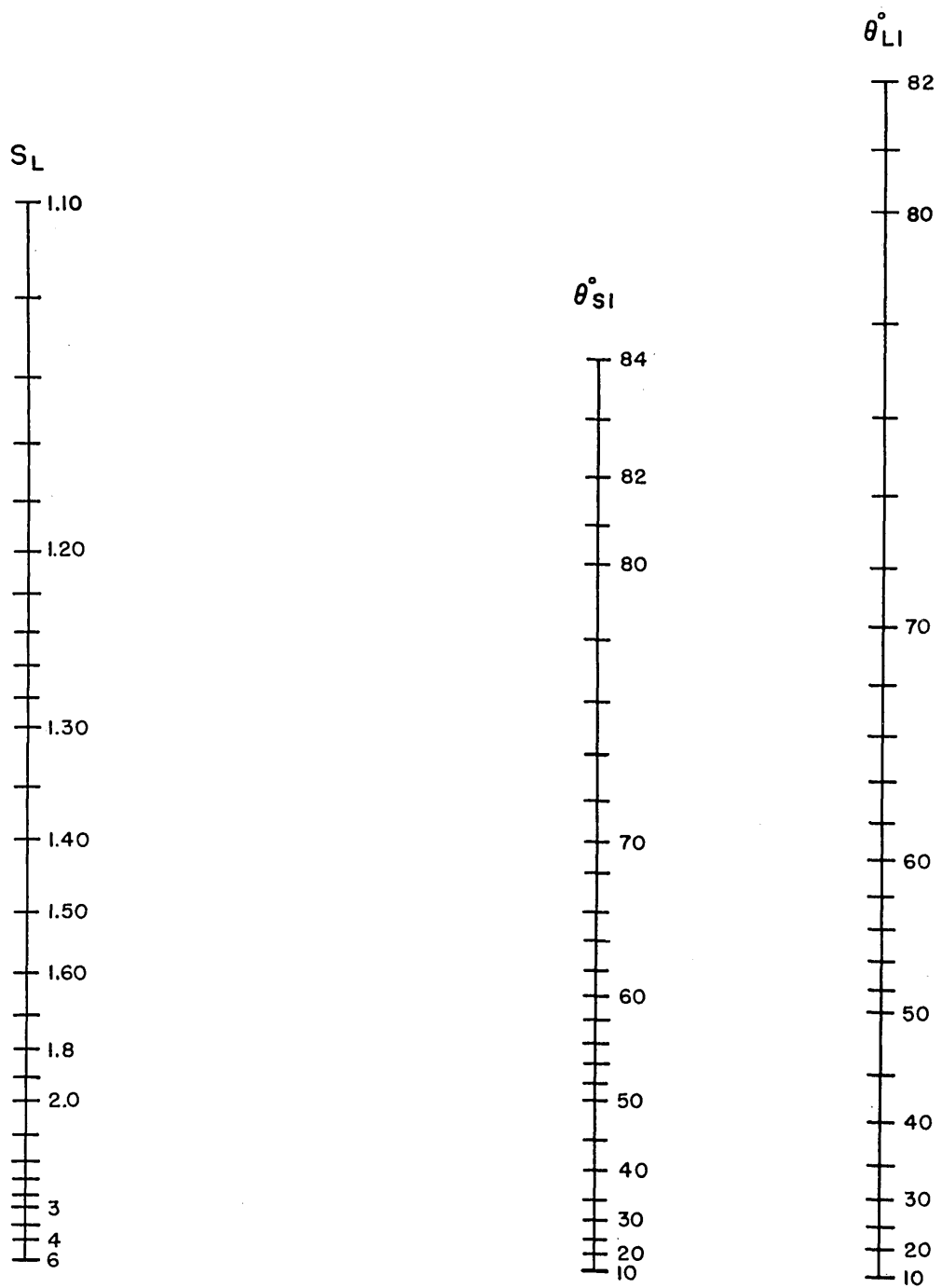

Plate 2

Nomograph for Method 5

18

S_L

1.10

1.20

1.30

1.40

1.50

1.60

1.8

2.0

3

4
6

θ°_{SI}

84

82

80

70

60

50

40

30

20
10

θ°_{LI}

82

80

70

60

50

40

30

20
10

Plate 3

Nomograph for Method 6
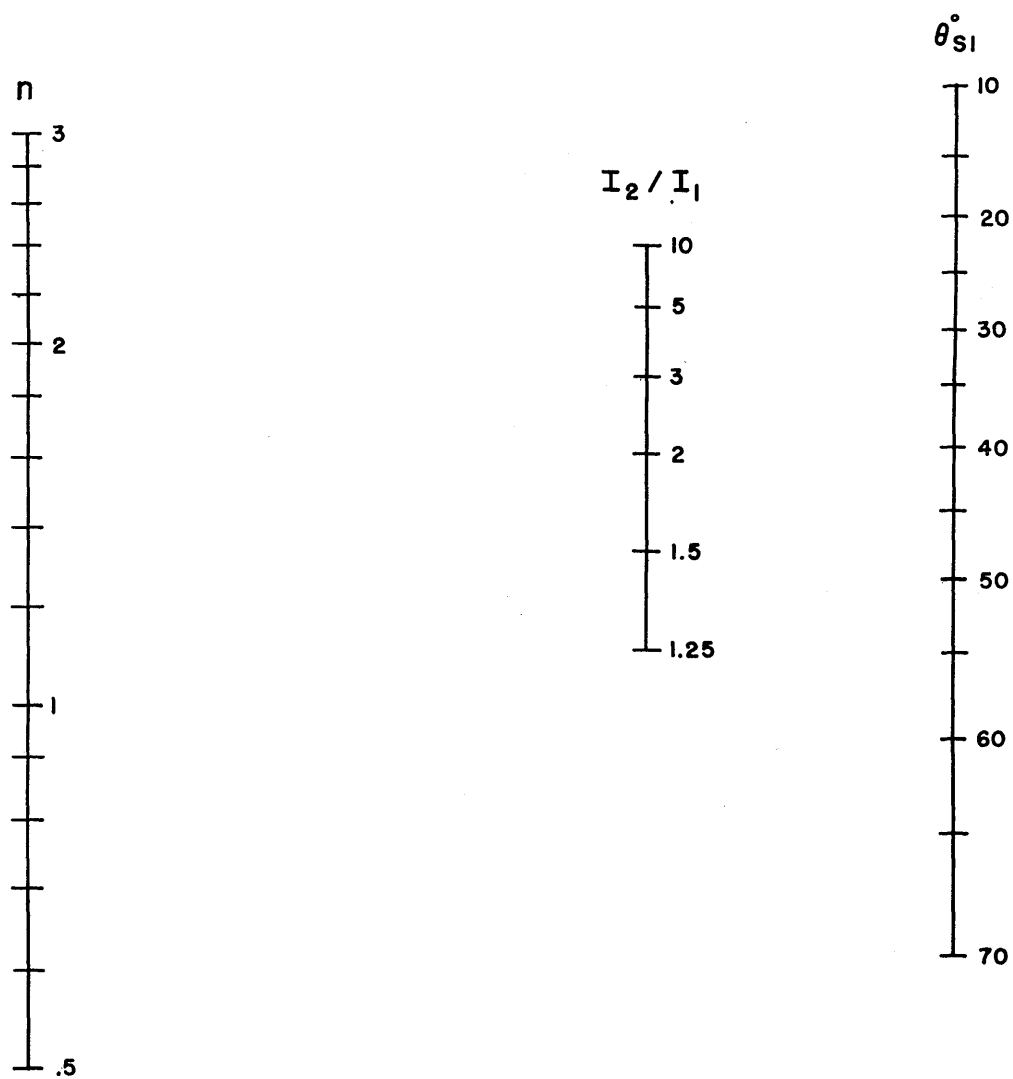
Plate 4

Nomograph giving law of detector for Methods 2, 5, 6, and 8

20

$db_L = 20 \log_{10} S_L$

θ_1°

θ_2°

60
50
40
30
20

1
5
10
20
30
40
50
60
70
80
85
89

6
4
3
2
1
.8
.6
.4
.2
.1
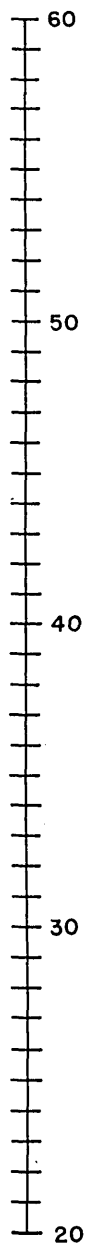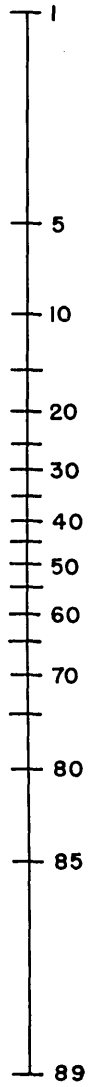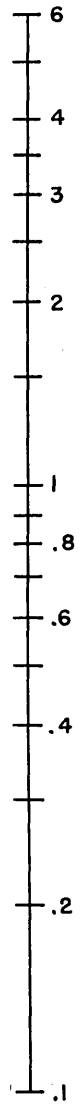
Plate 5

Nomograph for Method 10

$db_L = 20 \log_{10} S_L$

20

15

10

5

0

θ'°_{S2}

0
20
30
40
50
60
70
80
85

θ_{S1}

90
70
60
50
40
30
20
15
10
8
6
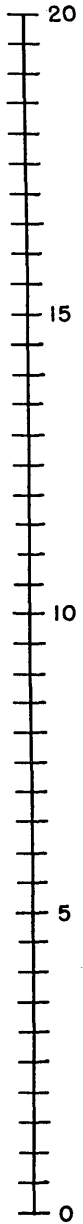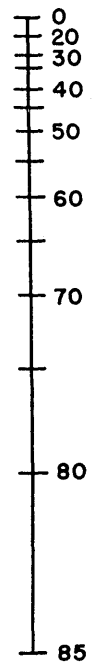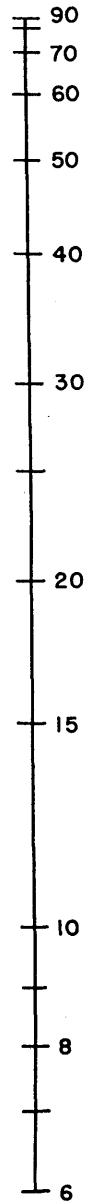
Plate 6

Nomograph for Method 13

APPENDIX I

Extension of the Methods for Obtaining the Voltage-Standing-Wave Ratio to Attenuating Transmission Lines

On a transmission line which has attenuation, a voltage distribution such as is shown in Figure 21(a) would be obtained on a loaded line. One similar to that shown in Figure 21(b) would be obtained on a shorted line. It is easily shown[1] that for the conditions of Figure 21(a) we have

$$K_L = K_L' \ \epsilon^{2\alpha X_L} \tag{85}$$
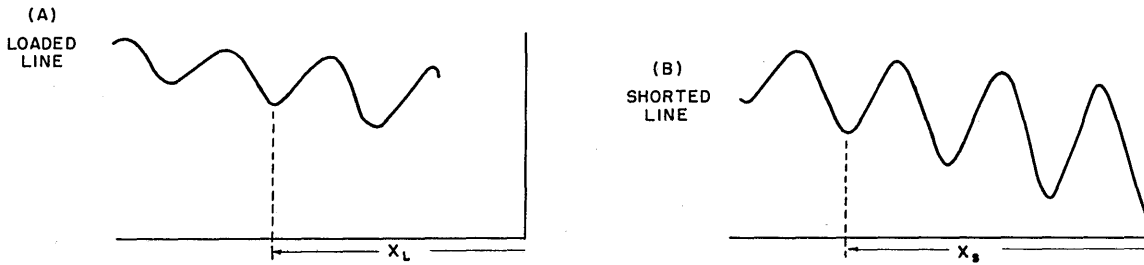
Fig. 21 - Voltage distributions obtained on attenuating transmission lines

where K_L is the true reflection coefficient of the load, K_L' is the reflection coefficient measured at distance X_L from the load and α is the line attenuation constant. In reference[1] it is also shown that for the case of Figure 21(b) we have

$$K_S' = \epsilon^{-2\alpha X_S} \tag{86}$$

where K_S' is the reflection coefficient measured at distance X_S from the load. Equation (86) is derived assuming a perfect shorting termination.

Multiplying equations (85) and (86) gives

$$K_L K_S' = K_L' \ \epsilon^{2\alpha (X_L - X_S)} \tag{87}$$

It is evident that to obtain K_L' or K_S' by using the value of V_{max} as obtained at one point and the value V_{min} as obtained at a point $\lambda/4$ from the maximum is not strictly correct. However values of V_{max} and V_{min} for both the loaded and shorted cases, can be obtained within a length of the line equal to $\lambda/2$. It is thus assumed that, although the line is

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[1] "The Solution of Transmission-Line Problems in the Case of Attenuating Transmission Line" - G. Glinski, Trans. AIEE, V. 65, p. 46, Feb. 1946

attenuating, the attenuation of the length of line equal to $\lambda/2$ is negligible, i.e. $\epsilon^{\alpha\lambda/2} \approx 1$. Equation (87) thus becomes

$$K_L \, K'_S = K'_L \qquad (88)$$

Substituting in equation (88) from equation (4) gives

$$S_L = \frac{S'_s \, S'_L - 1}{S'_s - S'_L} \qquad (89)$$

By means of equation (89) the value of VSWR at the load, S_L, can be obtained by measuring the VSWR on the shorted line at approximately the same point it was obtained on the loaded line.

If we assume that the attenuation of a length of line equal to $\lambda/2$ is negligible then adjacent maxima are practically equal. The voltage distribution within this region can then be expressed by the same voltage distribution used in the lossless line cases. Thus for the loaded line case we have

$$V_L^2 = V_{L\,max}^2 \, \sin^2 \theta_L + V_{L\,min}^2 \, \cos^2 \theta_L \qquad (90)$$

while for the shorted line we have

$$V_S^2 = V_{S\,max}^2 \, \sin^2 \theta_S + V_{S\,min}^2 \, \cos^2 \theta_S \qquad (91)$$

However these conditions are the same as those beginning with the last paragraph on page 12, where the effect of lossy shorting termination was to give a finite value of $V_{S\,min}$ on the shorted line. All the methods described in that chapter can be used here, however the quantity S_L appearing in those equations now becomes S'_L since we are obtaining the value of VSWR at some distance from the load. The quantity S'_S is obtained from equation (78). The values S'_L and S'_S are then substituted in equation (89).

For example consider the application of Method 13 to the case of attenuating transmission line. For that method we have

$$S'_L = \frac{\cos \theta'_{S2}}{\sin \theta_{S1}} \qquad (83)$$

and

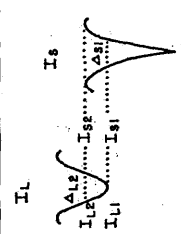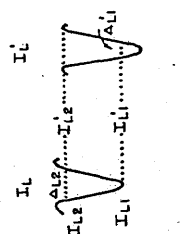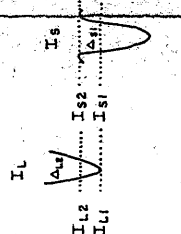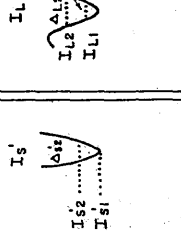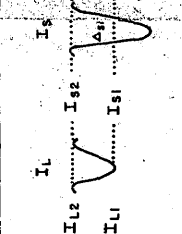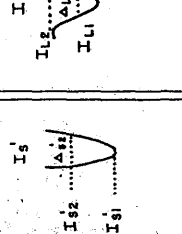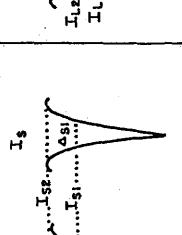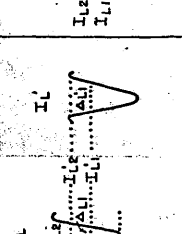$$S'_S = \frac{\cot \theta_{S1}}{\tan \theta'_{S2}} \qquad (78)$$
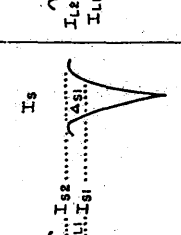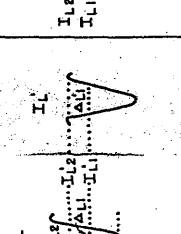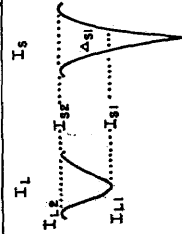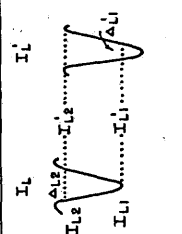
Substituting these into equation (89) gives

$$S_L = \frac{\cot \theta_{S1} \, \cos \theta'_{S2} - \tan \theta'_{S2} \, \sin \theta_{S1}}{\cos \theta_{S1} - \sin \theta'_{S2}} \qquad (92)$$

Using equation (92) with Method 13 automatically includes the effect of line attenuation.

* * *

TABLE I

SUMMARY OF METHODS FOR OBTAINING THE VOLTAGE-STANDING-WAVE RATIO

	LOAD MINIMA OBTAINABLE			LOAD MINIMA UNOBTAINABLE		
	LOSSLESS SHORT	NO SHORT USED	IMPERFECT SHORT	LOSSLESS SHORT	NO SHORT USED	IMPERFECT SHORT
HIGH VSWR $S_L > 10$	Fig. 8 $$S_L^2 - 1 = \frac{\cot^2 \theta_{S_1}}{\sin^2 \theta_{L_2}}$$ Eq. 44 See Plate 2	Fig. 15 $$S_L = \frac{\cot \theta_{L_1}'}{\tan \theta_{L_2}}$$ Eq. 65 See Plate 5	Fig. 18 $$S_L^2 = \frac{\dfrac{\cos^2 \theta_{S_1}'}{\sin^2 \theta_{S_1}} - \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_2}}$$ Eq. 81	Fig. 5 $$S_L^2 = \frac{\dfrac{\cos^2 \theta_{L_1}}{\sin^2 \theta_{S_1}} - \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_2} - \dfrac{\sin^2 \theta_{L_1}}{\sin^2 \theta_{S_1}}}$$ Eq. 35 See Plate 1	Fig. 14 $$S_L^2 = \frac{\cos^2 \theta_{L_1}' \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_1}' \sin^2 \theta_{L_2} - \sin^2 \theta_{L_2}}$$ Eq. 64	Fig. 17 $$S_L^2 = \frac{\dfrac{\cos^2 \theta_{S_1}'}{\sin^2 \theta_{S_1}} \cos^2 \theta_{L_1} - \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_2} - \dfrac{\cos^2 \theta_{S_1}'}{\sin^2 \theta_{L_2}} \sin^2 \theta_{L_1}}$$ Eq. 80
LOW VSWR $S_L < 10$	Fig. 11 $$S_L = \frac{1}{\sin \theta_{S_1}}$$ Eq. 48 See Fig. 12	Fig. 15 $$S_L = \frac{\cot \theta_{L_1}'}{\tan \theta_{L_2}}$$ Eq. 65	Fig. 20 $$S_L = \frac{\cos \theta_{S_1}'}{\sin \theta_{S_1}}$$ Eq. 83 See Plate 6	Fig. 9 $$S_L^2 = \frac{\cos^2 \theta_{L_1}}{\sin^2 \theta_{S_1} - \sin^2 \theta_{L_1}}$$ Eq. 46 See Plate 3	Fig. 14 $$S_L^2 = \frac{\cos^2 \theta_{L_1}' \cos^2 \theta_{L_2}}{\sin^2 \theta_{L_1}' \sin^2 \theta_{L_2} - \sin^2 \theta_{L_1}}$$ Eq. 64	Fig. 19 $$S_L^2 = \frac{\cos^2 \theta_{L_1}}{\dfrac{\sin^2 \theta_{S_1}}{\cos^2 \theta_{S_1}} - \sin^2 \theta_{L_1}}$$ Eq. 82